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Renormalisation group study of polyelectrolyte chains

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Abstract. The behaviour of long unscreened polyelectrolyte chains is studied using field-theoretic renormalisation group methods based on the renormalisation procedure of t'Hooft and Veltman. It is shown that a fixed point exists near dimension six and details are given of the calculation of the critical exponents up to $O(\varepsilon^2)$. It is also shown exactly that, as long as a fixed point exists, the radius of gyration R_G is given by $R_G \sim N^\nu$ with $\nu = 2/(d-2)$.

1. Introduction

Polyelectrolytes are macromolecules with ionisable groups that in solution partially dissociate to give polyions and oppositely charged counterions (Oosawa 1971, Katchalsky 1953). As is well known (Oosawa 1971, de Gennes *et al* 1976, Pfeuty *et al* 1977, Pfeuty 1978), charged polymers have a much larger size than uncharged excluded-volume ones, this being a consequence of the long range of the (unscreened) Coulomb potential.

Nevertheless, when salt or other electrolyte is added in high enough concentration to the solution, the Coulomb interaction is screened and the polymer chain recoils on itself. These electrostatic effects on the conformation of charged chains in natural or synthetic polyelectrolyte solutions are believed to be responsible for many important phenomena. A good understanding of the conformation properties of these systems is thus required and, as a starting point, the evaluation of the polyion size in the dilute regime will be considered in this paper.

In the limit of very low concentration it is assumed that each chain behaves independently of the others. In this limit, phenomenological calculations of the Flory type (de Gennes *et al* (1976) and references therein) show that the radius of gyration R_G of an isolated polyion grows with the polymerisation index N proportionally to N itself. More generally, for a chain in a d-dimensional space interacting with a generalised Coulomb potential given by

$$V(R) = Q^2/\varepsilon R^{d-2}$$

R_G is believed to grow as $N^{3/d}$.

More rigorous calculations can be performed, starting from a microscopic model for the chain configuration probability distribution (Edwards 1966) and making use of field theory and renormalisation group (RG) methods. These techniques allow us to study the asymptotic ($N \to \infty$) behaviour of the chain configuration functions, and thus they represent a way to calculate its size exponent ν, defined such that

$$\lim_{N\to\infty} R_G \sim N^\nu.$$

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In this work we present a detailed calculation of the exponent ν for an isolated polyelectrolyte in a d-dimensional space. The Lagrangian theory of a model system in which every monomer is carrying a fixed charge and the counterions are uniformly distributed in the solution is discussed. The RG treatment for this system is then shown to give an exact result for the exponent ν, valid at least near dimension six.

This same problem has already received attention and has been tackled with similar techniques by Pfeuty *et al* (1977). In their letter, these authors successfully calculate the exponent ν for the isolated polyelectrolyte in d dimensions using a field-theoretic RG method, the details of which, however, have never been published. In their paper they find the same results for the exponents η and ν to order ε^2 as we do, although they propose that the result for ν should be true to all orders in ε and give phenomenological arguments to support this proposal.

The present paper thus completes the work by Pfeuty *et al*, although it is mainly of technical interest, as no new evidence is provided for the value $\nu = 1$, not yet rigorously proved, of the size exponent of the polyelectrolyte chain in three dimensions.

The paper is organised as follows. In § 2 the microscopic model is presented and its equivalent Lagrangian field theory introduced through the use of the celebrated $n = 0$ equivalence theorem (de Gennes 1972, des Cloizeaux 1975), of which a new direct proof is given in appendix 1. The Lagrangian field theory, a ϕ^4 model with a two-point Coulomb interaction, is a non-trivially renormalisable one. It is thus transformed into a new point-interacting field theory through the introduction of a new (Coulomb) scalar field.

The perturbative renormalisation theory for the new model is presented in § 3 for the case of general n. In § 4, the critical exponents η and ν for the polymer ($n = 0$) problem are calculated to order ε^2, using t'Hooft and Veltman's (1972) dimensional regularisation and renormalisation by minimal subtraction of singularities. In this particular RG approach we recover the results of Pfeuty *et al* and rigorously prove that the result for ν holds to every order in ε. Moreover, we show clearly that the validity of the result depends upon the existence of a non-trivial, stable fixed point in d dimensions. Near dimension six, such a fixed point is found by standard ε-expansion treatment. For a general value of n, our method leads to a scaling relation for two critical exponents of the Lagrangian field theory, whose physical meaning remains unknown.

Finally, in appendix 3 the renormalisation of the Coulomb field theory is presented and the exponents η and ν recalculated in this framework to order ε^2, reaching results consistent with the previous calculation.

2. Definition of the model

We consider an isolated polyion in a good solvent, assuming that on each monomer there is a charge Ze. The Coulomb interaction between pairs of monomers will be considered dominant and, in this first attempt to calculate the size of the polyion, we will assume that the distribution of counterions in the region occupied by the polyion is uniform (de Gennes *et al* 1976, Pfeuty *et al* 1977, Pfeuty 1978).

Following the usual procedure (Edwards 1966, des Cloizeaux 1975), we associate with each point of the continuous chain a vector $\boldsymbol{r}(s)$, where s is the length, measured along the chain, between the point defined by $\boldsymbol{r}(s)$ and one end of the chain ($0 \leqslant s \leqslant L$, $L = Nl$ being the chain length and l the link length). The statistical weight relative to a

configuration $\mathscr{C} = \{\boldsymbol{r}(s),\ 0 \leqslant s \leqslant L\}$ of the polyion will be given by

$$P(\mathscr{C}, L) = Z^{-1}(L) \exp[-H(\mathscr{C})] \tag{2.1}$$

with

$$H(\mathscr{C}) = H_0 + H_I = \frac{3}{2l}\int_0^L \mathrm{d}s \left(\frac{\mathrm{d}\boldsymbol{r}(s)}{\mathrm{d}s}\right)^2 + \frac{\beta}{2l^2}\int_0^L \mathrm{d}s' \int_0^L \mathrm{d}s'' \int \mathrm{d}\boldsymbol{q}\, v(\boldsymbol{q}) \exp\{-\mathrm{i}\boldsymbol{q}\cdot[\boldsymbol{r}(s') - \boldsymbol{r}(s'')]\}, \tag{2.2}$$

the partition function being defined as

$$Z(L) = \int D\mathscr{C} \exp[-H(\mathscr{C})], \qquad D\mathscr{C} \equiv \prod_{s=0}^{L} \mathrm{d}\boldsymbol{r}(s). \tag{2.3}$$

The function $v(\boldsymbol{q})$ in (2.2) is the Fourier transform of the bare d-dimensional Coulomb potential, $V(r) = Z^2 e^2/\varepsilon r^{d-2}$, where the $\boldsymbol{q} = \boldsymbol{0}$ term has been cancelled by the interaction term in H_I due to the uniform counterion background. That is:

$$v(\boldsymbol{q}) = \begin{cases} 4\pi Z^2 e^2/\varepsilon q^2, & \boldsymbol{q} \neq \boldsymbol{0}, \\ 0, & \boldsymbol{q} = \boldsymbol{0}. \end{cases} \tag{2.4}$$

The integral $\int \mathrm{d}\boldsymbol{q} \equiv \int \mathrm{d}^d q/(2\pi)^d$ is a multiple integral over a d-dimensional momentum space in which a natural cut-off $|\boldsymbol{q}| \leqslant \Lambda \simeq l^{-1}$ is introduced. Finally, $\beta = (k_B T)^{-1}$. The quantity of interest here is the probability distribution for the chain ends, defined as (des Cloizeaux 1975)

$$G(\boldsymbol{x}_1, \boldsymbol{x}_2; L) = \frac{\int D\mathscr{C}\, \delta[\boldsymbol{r}(0) - \boldsymbol{x}_1]\delta[\boldsymbol{r}(L) - \boldsymbol{x}_2]\, \mathrm{e}^{-H(\mathscr{C})}}{\int D\mathscr{C}\, \mathrm{e}^{-H_0(\mathscr{C})}} \tag{2.5}$$

and satisfying the property of invariance under space translations:

$$G(\boldsymbol{x}_1, \boldsymbol{x}_2; L) = G(\boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{0}; L) = G(\boldsymbol{0}, \boldsymbol{x}_2 - \boldsymbol{x}_1; L).$$

The mean-square end-to-end distance is, for instance, given in terms of G:

$$\langle R^2 \rangle = \int \mathrm{d}\boldsymbol{R}\, R^2 G(\boldsymbol{R}, L) \Big/ \int \mathrm{d}\boldsymbol{R}\, G(\boldsymbol{R}, L), \tag{2.6}$$

and we will be concerned with the calculation of the size exponent for the polyelectrolyte chain, defined by the asymptotic behaviour

$$\lim_{L\to\infty} \langle R^2 \rangle \sim L^{2\nu}. \tag{2.7}$$

To do so, we will use the equivalence, first established by de Gennes (1972), between the polymer problem and a zero-component magnetic system near its critical point. In the limit $L \to \infty$ the polymer is a critical object and its size exponent is formally equal to the critical exponent ν for the $n = 0$ magnetic system.

More precisely, the Laplace transform of the probability (2.5),

$$G(\boldsymbol{x}_1, \boldsymbol{x}_2; s) = \int_0^\infty \frac{\mathrm{d}L}{l}\, \mathrm{e}^{-sL/l}\, G(\boldsymbol{x}_1, \boldsymbol{x}_2; L), \tag{2.8}$$

is formally equivalent to the connected two-point Green function $G_c^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ in a

zero-component field theory with Lagrangian

$$\mathscr{L}[\boldsymbol{\phi}] = \int \mathrm{d}\boldsymbol{x}\{\tfrac{1}{2}r_0[\boldsymbol{\phi}(\boldsymbol{x})]^2 + \tfrac{1}{2}[\nabla\boldsymbol{\phi}(\boldsymbol{x})]^2\} + \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}\boldsymbol{y}[\boldsymbol{\phi}(\boldsymbol{x})]^2 V(\boldsymbol{x}-\boldsymbol{y})[\boldsymbol{\phi}(\boldsymbol{y})]^2 \tag{2.9}$$

where $\boldsymbol{\phi}(\boldsymbol{x})$ is a real $n \to 0$ component field, $V(\boldsymbol{x})$ is the interaction potential and $r_0 = sl^{-2}$.

This equivalence was first proved by using perturbation expansion (des Cloizeaux 1975, de Gennes 1972) and very recently by direct analytic methods (Orland 1980, Duplantier 1980); a new direct method is however reported in appendix 1. See also Daoud *et al* (1975) for evidence of the magnetic analogy of the polymer problem on more physical grounds. Hence, we will study, in what follows, the critical behaviour of a system described, in a momentum space where the cut-off Λ is introduced, by the Lagrangian

$$\mathscr{L}[\boldsymbol{\phi}] = \tfrac{1}{2}\sum_{i=1}^{n} \int^{\Lambda} \mathrm{d}\boldsymbol{k}\ (r_0 + k^2)\phi_i(\boldsymbol{k})\phi_i(-\boldsymbol{k}) + \sum_{i,j=1}^{n} \int^{\Lambda} \mathrm{d}\boldsymbol{k}_1 \ldots \mathrm{d}\boldsymbol{k}_4 \phi_i(\boldsymbol{k}_1)\phi_i(\boldsymbol{k}_2)v(\boldsymbol{k}_1+\boldsymbol{k}_2)\phi_j(\boldsymbol{k}_3)\phi_j(\boldsymbol{k}_4)\delta(\boldsymbol{k}_1+\ldots+\boldsymbol{k}_4) \tag{2.10}$$

in the limit $n \to 0$.

To do so, we will use the field-theoretic RG method in the approach of the dimensional regularisation and renormalisation by minimal subtraction of singularities developed by t'Hooft and Veltman in relativistic quantum field theory (t'Hooft and Veltman 1972) and successfully introduced by Amit in statistical physics (Amit 1976) (see however Amit (1978) for a general, detailed and elegant introduction to RG methods). This particular approach represents a natural way of applying field theory to critical phenomena, and is technically advantageous as it provides many internal checks as the calculation proceeds.

It is, however, not convenient to tackle, with RG methods, the critical behaviour of model (2.10) directly, since (as we will show in appendix 3) the Coulomb-interaction theory is renormalisable in an unusual way, as the interaction is not a point one. We find it more convenient to transform the theory (2.10) into another one with point interaction, to which almost standard techniques can be applied, by introducing a new field. This new scalar field $\psi(\boldsymbol{x})$ is defined such that, apart from normalisation factors,

$$Z = \int D\boldsymbol{\phi}\ \exp(-\mathscr{L}[\boldsymbol{\phi}]) = \int D\boldsymbol{\phi} D\psi\ \exp(-L[\boldsymbol{\phi}, \psi]). \tag{2.11}$$

The new Lagrangian will be given by

$$L[\boldsymbol{\phi}, \psi] = \tfrac{1}{2}\sum_{i=1}^{n} \int^{\Lambda} \mathrm{d}\boldsymbol{k}(r_0 + k^2)\phi_i(\boldsymbol{k})\phi_i(-\boldsymbol{k}) + \int^{\Lambda} \mathrm{d}\boldsymbol{k}\,\omega(\boldsymbol{k})\psi(\boldsymbol{k})\psi(-\boldsymbol{k}) + \sum_{i=1}^{n} \int^{\Lambda} \mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3 \frac{g_0}{3!}\phi_i(\boldsymbol{k}_1)\phi_i(\boldsymbol{k}_2)\psi(\boldsymbol{k}_3)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3), \tag{2.12}$$

where g_0 and $\omega(\boldsymbol{k})$ are defined by

$$\omega(-\boldsymbol{k}) = \omega(\boldsymbol{k}) \qquad \text{and} \qquad (g_0/3!)^2 = -4\omega(\boldsymbol{k})v(\boldsymbol{k}). \tag{2.13}$$

As the Coulomb interaction is

$$v(\boldsymbol{k}) = \begin{cases} (\lambda_0/4!)k^{-2}, & \boldsymbol{k} \neq \boldsymbol{0} \\ 0, & \boldsymbol{k} = \boldsymbol{0}, \end{cases} \qquad \frac{\lambda_0}{4!} = 18\,\pi\beta\frac{Z^2e^2}{\varepsilon l^4}, \tag{2.14}$$

we can choose

$$\omega(\boldsymbol{k}) = \begin{cases} \tfrac{1}{2}k^2, & \boldsymbol{k} \neq \boldsymbol{0} \\ 0, & \boldsymbol{k} = \boldsymbol{0}, \end{cases} \qquad g_0^2 = -3\lambda_0, \tag{2.15}$$

giving

$$L[\boldsymbol{\phi}, \psi] = \tfrac{1}{2}\sum_{i=1}^{n}\int^{\Lambda} \mathrm{d}\boldsymbol{k}\ (r_0 + k^2)\phi_i(\boldsymbol{k})\phi_i(-\boldsymbol{k}) + \tfrac{1}{2}\int^{\Lambda} \mathrm{d}\boldsymbol{k}\, k^2\psi(\boldsymbol{k})\psi(-\boldsymbol{k})$$
$$+\frac{1}{3!}g_0\sum_{i=1}^{n}\int^{\Lambda} \mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\phi_i(\boldsymbol{k}_1)\phi_i(\boldsymbol{k}_2)\psi(\boldsymbol{k}_3)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) \tag{2.16}$$

representing a point $\phi^2\psi$-theory, as it can be written in the form

$$L[\boldsymbol{\phi}, \psi] = \int \mathrm{d}\boldsymbol{x}\{\tfrac{1}{2}r_0[\boldsymbol{\phi}(\boldsymbol{x})]^2 + \tfrac{1}{2}[\nabla\boldsymbol{\phi}(\boldsymbol{x})]^2 + \tfrac{1}{2}[\nabla\psi(\boldsymbol{x})]^2 + (1/3!)g_0[\boldsymbol{\phi}(\boldsymbol{x})]^2\psi(\boldsymbol{x})\}. \tag{2.17}$$

We remark that the choice (2.15) for g_0 can generate imaginary fixed points g^* under the RG transformation, as we will in fact see in § 4. This is however an artifact of the transformation, as relevant quantities are given in terms of a series of powers of λ and hence of even powers of g, leading to real values.

3. Perturbation expansion and renormalisation of the theory

3.1. Feynman diagrams for the relevant irreducible vertices

The Lagrangian (2.16) and (2.17) can be regarded as an $(n+1)$-vector ϕ^3-field theory Lagrangian by introducing the $(n+1)$-component field

$$\hat{\boldsymbol{\phi}} = (\phi_1, \ldots, \phi_n, \psi)$$

with masses

$$r_{01} = r_0, \qquad r_{02} = r_0, \ldots, \qquad r_{0n} = r_0, \qquad r_{0n+1} = 0,$$

and rewriting the Lagrangian in the form

$$L[\hat{\boldsymbol{\phi}}] = \int \mathrm{d}\boldsymbol{x}\left(\tfrac{1}{2}\sum_{i=1}^{n+1} r_{0i}[\hat{\phi}_i(\boldsymbol{x})]^2 + \tfrac{1}{2}\sum_{i=1}^{n+1}[\nabla\hat{\phi}_i(\boldsymbol{x})]^2 + \frac{1}{3!}g_0\sum_{ijk=1}^{n+1} Q_{ijk}\hat{\phi}_i(\boldsymbol{x})\hat{\phi}_j(\boldsymbol{x})\hat{\phi}_k(\boldsymbol{x})\right), \tag{3.1}$$

where

$$Q_{ijk} = \tfrac{1}{3}[\delta_{ij}(1-\delta_{i,n+1})\delta_{k,n+1} + 2 \text{ permutations}]$$

is a tensor symmetric in all three indices.

Green functions can be defined by

$$G^{(N,L)}_{i_1\ldots i_N}(k_1\ldots k_N, p_1\ldots p_L) = \frac{1}{2^L}\langle\hat{\phi}_{i_1}(k_1)\ldots\hat{\phi}_{i_N}(k_N)\boldsymbol{\phi}^2(p_1)\ldots\boldsymbol{\phi}^2(p_L)\rangle$$

and their perturbation expansions written down as power series in the coupling constant

g_0, each particular contribution being represented by a Feynman diagram, drawn and calculated according to the usual rules (Amit 1978).

The bare propagators are

$$G_{0i}(k) = 1/(r_{0i} + k^2)$$

and it turns out in the end that only two of them are distinct, representing the bare propagators for the fields $\boldsymbol{\phi}$ and ψ:

$$G_0^{\phi_i}(k) = \frac{1}{r_0 + k^2}, \qquad G_0^{\psi}(k) = \frac{1}{k^2} \qquad (=0 \text{ if } \boldsymbol{k} = \boldsymbol{0}). \tag{3.2}$$

Here, they will be represented diagrammatically by a single full line and by a single broken line, respectively.

Drawing and evaluating diagrams is easier if the interaction vertex is represented by a point at the meeting of two single full lines, carrying momentum labels $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ and the same spin label i, and a single broken line carrying momentum label $\boldsymbol{k}_3$. In drawing graphs we must then remember that only lines of the same species can be contracted into pairs, and in evaluating them we include a factor $-(1/3!)\, g_0 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)$ for every interaction vertex. Combinatorial and symmetry factors will be calculated in the usual way, taking into account the above remarks.

As we are interested in the $n = 0$ limit of the field theory, loop diagrams will be excluded from the perturbation expansions in the main calculations.

In this limit, non-vanishing Green functions of interest are

$$\begin{aligned}
&G^{(2)}_{\phi\phi}(\boldsymbol{k}, -\boldsymbol{k}) = \langle \phi_i(\boldsymbol{k}) \phi_i(-\boldsymbol{k}) \rangle, \\
&G^{(2)}_{\psi\psi}(\boldsymbol{k}, -\boldsymbol{k}) = \langle \psi(\boldsymbol{k}) \psi(-\boldsymbol{k}) \rangle \underset{n \to 0}{\to} \frac{1}{k^2}, \\
&G^{(2,1)}_{\phi\phi}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{p}) = \langle \phi_i(\boldsymbol{k}_1) \phi_i(\boldsymbol{k}_2) \tfrac{1}{2} \boldsymbol{\phi}^2(\boldsymbol{p}) \rangle, \\
&G^{(3)}_{\phi\phi\psi}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \langle \phi_i(\boldsymbol{k}_1) \phi_i(\boldsymbol{k}_2) \psi(\boldsymbol{k}_3) \rangle.
\end{aligned} \tag{3.3}$$

The one-particle-irreducible (1PI) parts of any Green function can still be obtained from a generating functional $\Gamma[\bar{\boldsymbol{\phi}}, \bar{\psi}, t]$. This is defined through a Legendre transform of the generating functional $F[\boldsymbol{h}, j, t]$ for the connected parts of $G^{(N,L)}$, which is in turn constructed from the Lagrangian (2.17), adding a source term of the form

$$-\int \mathrm{d}\boldsymbol{x} \{\boldsymbol{h}(\boldsymbol{x}) \cdot \boldsymbol{\phi}(\boldsymbol{x}) + j(\boldsymbol{x}) \psi(\boldsymbol{x}) + \tfrac{1}{2} t(\boldsymbol{x}) [\boldsymbol{\phi}(\boldsymbol{x})]^2\}. \tag{3.4}$$

The transform Γ is defined by

$$\Gamma[\bar{\boldsymbol{\phi}}, \bar{\psi}, t] + F[\boldsymbol{h}, j, t] = \int \mathrm{d}\boldsymbol{x} [\boldsymbol{h}(\boldsymbol{x}) \cdot \bar{\boldsymbol{\phi}}(\boldsymbol{x}) + j(\boldsymbol{x}) \bar{\psi}(\boldsymbol{x})], \tag{3.5}$$

with

$$\begin{cases} \bar{\phi}_i(\boldsymbol{x}) = \delta F / \delta h_i(\boldsymbol{x}), \\ \bar{\psi}(\boldsymbol{x}) = \delta F / \delta j(\boldsymbol{x}), \end{cases} \qquad \begin{cases} h_i(\boldsymbol{x}) = \delta \Gamma / \delta \bar{\phi}_i(\boldsymbol{x}), \\ j(\boldsymbol{x}) = \delta \Gamma / \delta \bar{\psi}(\boldsymbol{x}). \end{cases} \tag{3.6}$$

By successive differentiations of equation (3.5), we can still prove that the relationship

between connected Green functions $C^{(N,L)}$ and their 1PI parts $\Gamma^{(N,L)}$ is given by

$$\begin{cases} C^{(N,L)}_{i_1\ldots i_N}(k_1\ldots k_N; p_1\ldots p_L) = -\sum_{j_1\ldots j_N} C^{(2)}_{i_1 j_1}(k_1)\ldots C^{(2)}_{i_N j_N}(k_N)\Gamma^{(N,L)}_{j_1\ldots j_N}(k_1\ldots k_N, p_1\ldots p_L), \\ C^{(2,L)}_{i_1 i_2}(k_1, k_2; p_i) = \sum_{j_1 j_2} C^{(2)}_{i_1 j_1}(k_1) C^{(2)}_{i_2 j_2}(k_2)\Gamma^{(2,L)}_{j_1 j_2}(k_1, k_2; p_i), \end{cases} \tag{3.7}$$

where the indices i_α and j_α refer to components of the $\hat{\boldsymbol{\phi}} = (\boldsymbol{\phi}, \psi)$ field.

We can at this point draw the diagrammatic expansions for the 1PI parts of the Green functions (3.3) in the $n = 0$ limit; to order two-loops, these are shown in figure 1(a). The diagrammatic expansion, to order two-loops, for the vertex $\Gamma^{(2)}_{\psi\psi}(k)$ is drawn in figure 1(b) for the general n case.

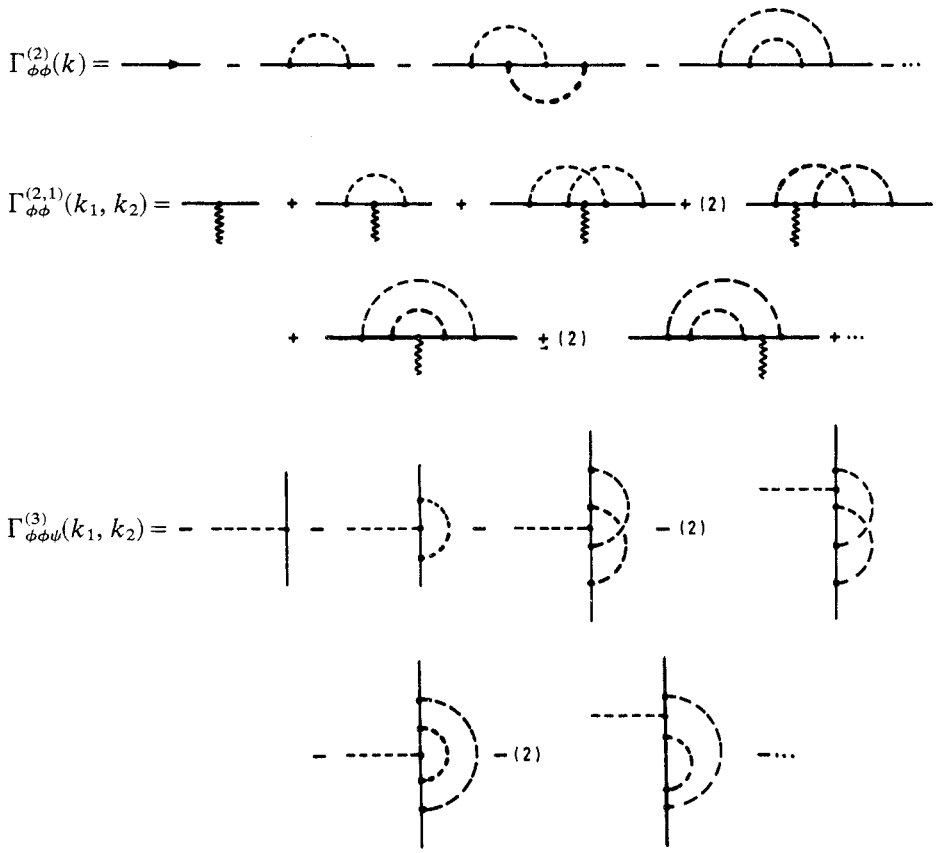

Figure 1. (a) Two-loops order diagrammatic expansions of $\Gamma^{(2)}_{\phi\phi}$, $\Gamma^{(2,1)}_{\phi\phi}$, $\Gamma^{(3)}_{\phi\phi\psi}$ in the $n = 0$ limit.

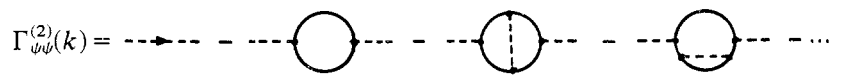

Figure 1. (b) Two-loops order diagrammatic expansion for $\Gamma^{(2)}_{\psi\psi}$ (the 1PI part of the dressed interaction) for general n.

As usual, the wiggly line ⇝ represents a ϕ^2 insertion and the numbers reported on certain graphs represent their multiplicity numbers, when different from one.

We remark that in all the diagrams drawn for the 1PI parts the contributions of the external legs must not be taken into account, and we stress the fact that the term 1PI here refers both to ϕ- and to ψ-body cuts, as it should from definition (3.5) of the generating functional Γ. This completely irreducible character of the vertex functions $\Gamma^{(N,L)}$ for the present theory will be partially lost for the Coulomb interaction theory (see appendix 3).

3.2. Primitive divergences and critical dimensionality

We start by illustrating our renormalisation technique.

As explained by Amit (1976, 1978), we can learn about the critical behaviour of our theory by looking at the ultraviolet divergences of the momentum integrals in our Feynman diagrams when the cut-off Λ is extended artificially to infinity, this being the essence of the dimensional regularisation procedure. The integrals, with no cut-off, are calculated for values of the dimensionality d for which they are finite and then analytically continued in all the complex plane spanned by d. Ultraviolet singularities then appear as poles of all orders in the variable $d_c - d = \varepsilon$, where d_c is the critical dimensionality of the problem. To determine d_c we look at the primitive UV divergence of a general Feynman graph of a vertex function with E external legs, r interaction vertices, I internal lines and L loops (all of unspecified origin). Then we have: primitive UV divergence $\sim\Lambda^{\delta}$, where

$$\delta = Ld - 2I, \qquad L = I - (r-1), \qquad I = \tfrac{1}{2}(3r - E),$$

so that

$$\delta = -r\delta_3 + (d + E - \tfrac{1}{2}Ed), \qquad \delta_3 = 3 - \tfrac{1}{2}d. \tag{3.8}$$

In field-theoretic language, d_c is defined such that for $d \leqslant d_c$ the power of the UV divergence, δ, does not increase with the order r of the diagram. This entails $\delta_3 \geqslant 0$, and if we write $\delta_3 = \frac{1}{2}(d_c - d)$, then from (3.8) we find

$$d_c = 6, \tag{3.9}$$

which means that our expansion parameter has to be

$$\varepsilon = 6 - d. \tag{3.10}$$

We remark that the critical number of dimensions for the polyelectrolyte, found here by field-theoretic arguments, agrees with the value proposed by de Gennes (de Gennes *et al* 1976) on more physical grounds.

3.3. Renormalisation of the theory

According to the general renormalisation theory, our theory is renormalisable for $d \leqslant 6$, when only a finite number of parameters is needed to convert the theory to one with all vertex functions finite at every order.

To find out the primitively divergent vertex functions, we observe that at the critical dimensionality $d = d_c$ the power of the UV divergence of a vertex $\Gamma^{(E,L)}$ becomes, from (3.8),

$$\delta_c - 2L = 6 - 2E - 2L, \tag{3.11}$$

where L is now the number of ϕ^2 insertions.

Thus, primitively UV divergent vertex functions will be those for which $6-2E-2L \geqslant 0$. In the limit $n=0$, only the functions $\Gamma^{(2)}$, $\Gamma^{(2,1)}$ and $\Gamma^{(3)}$ must be renormalised. To find out how the renormalisation has to be done, we first confine our discussion to the critical theory, $r_0 = r_{0c}$, in which the renormalised 'mass' r is zero. As is well known, this situation corresponds to the case of an infinitely long polymer (des Cloizeaux 1975), for which the relationship $N \sim (r_0 - r_{0c})^{-1}$ is expected.

We have then to introduce into the problem an arbitrary momentum scale, κ, so that the bare and renormalised coupling constants can be written in terms of κ and two dimensionless expansion parameters:

$$g_0 = w_0 \kappa^{\varepsilon/2}, \qquad g = w\kappa^{\varepsilon/2}. \tag{3.12}$$

Together with the dimensionless renormalisation function $w_0(w, \varepsilon)$ we introduce the vector of parameters

$$\boldsymbol{Z}_{\hat{\boldsymbol{\phi}}}(w, \varepsilon) = (Z_{\phi_1}, Z_{\phi_2}, \ldots, Z_{\phi_n}, Z_\psi), \tag{3.13}$$

representing the field renormalisation functions, and a last renormalisation parameter representing the $\boldsymbol{\phi}^2$-renormalisation function $Z_{\phi^2}(w, \varepsilon)$.

We then expect the renormalisation prescription to be of the form

$$\Gamma^{(N,L)}_{i_1 i_2 \ldots i_N \mathrm{R}}(k_i; p_i; w, \varepsilon, \kappa) = Z_{i_1}^{1/2} Z_{i_2}^{1/2} \ldots Z_{i_N}^{1/2} Z_{\phi^2}^{L} \Gamma^{(N,L)}_{i_1 i_2 \ldots i_N}(k_i; p_i; g_0), \tag{3.14}$$

the index i_α referring to one of the $\boldsymbol{\phi}$- components or to ψ. However, since all the masses of the components of the field $\boldsymbol{\phi}$ are equal to r_0 in the bare theory, we expect that only two components of $\boldsymbol{Z}_{\hat{\boldsymbol{\phi}}}$ are distinct, i.e. we must have

$$Z_{\phi_i} = Z_\phi \qquad \text{for } i = 1, 2, \ldots, n; \qquad Z_\psi \neq Z_\phi.$$

To check that we are using the correct renormalisation procedure, we put (3.14) in terms of counter terms. Since we have

$$C^{(2)}_{i_1 i_2 \mathrm{R}}(k_1, k_2) = Z_{i_1}^{-1/2} Z_{i_2}^{-1/2} C^{(2)}_{i_1 i_2}(k_1, k_2), \tag{3.15}$$

as we can see from $\Sigma_j C^{(2)}_{i_1 j \mathrm{R}}(k) \Gamma^{(2)}_{j i_2 \mathrm{R}}(k) = \delta_{i_1 i_2}$ using (3.14) for $\Gamma^{(2)}_{j i_2 \mathrm{R}}(k)$, the renormalisation prescription for the connected Green functions, as derived from insertion of (3.14) and (3.15) into (3.7), will be

$$C^{(N,L)}_{i_1 \ldots i_N \mathrm{R}}(k_i; p_i; w, \varepsilon, \kappa) = Z_{i_1}^{-1/2} \ldots Z_{i_N}^{-1/2} Z_{\phi^2}^{L} C^{(N,L)}_{i_1 \ldots i_N}(k_i; p_i; g_0). \tag{3.16}$$

Hence, the renormalised field theory may be derived from a new Lagrangian, which in the $(n+1)$-component field formalism will be given by

$$L'[\hat{\boldsymbol{\phi}}, \boldsymbol{h}, t] = L[\hat{\boldsymbol{\phi}}] - \int \mathrm{d}\boldsymbol{x} \left(\sum_{i=1}^{n+1} h'_i(\boldsymbol{x}) \hat{\phi}_i(\boldsymbol{x}) + \tfrac{1}{2} t'(\boldsymbol{x}) [\boldsymbol{\phi}(\boldsymbol{x})]^2 \right), \tag{3.17}$$

with $L[\hat{\boldsymbol{\phi}}]$ given by (3.1) and

$$h'_i(\boldsymbol{x}) = Z_i^{-1/2} h_i(\boldsymbol{x}), \qquad t'(\boldsymbol{x}) = Z_{\phi^2} t(\boldsymbol{x}) \tag{3.18}$$

new sources.

Redefining the field components according to

$$\hat{\phi}_{i\mathrm{R}}(\boldsymbol{x}) = Z_i^{-1/2} \hat{\phi}_i(\boldsymbol{x}), \tag{3.19}$$

the Lagrangian (3.17) can be written as

$$L'[\hat{\boldsymbol{\phi}}, \boldsymbol{h}, t] = L_{\mathrm{R}}[\hat{\boldsymbol{\phi}}_{\mathrm{R}}] - \int \mathrm{d}\boldsymbol{x}\left(\sum_i h_i \hat{\phi}_{i\mathrm{R}} + \tfrac{1}{2} t \boldsymbol{\phi}_{\mathrm{R}}^2\right) + \mathrm{CT}, \tag{3.20}$$

where

$$L_{\mathrm{R}}[\hat{\boldsymbol{\phi}}] = \int \mathrm{d}\boldsymbol{x}\left(\frac{1}{2}\sum_i r_i \hat{\phi}_i^2 + \frac{1}{2}\sum_i (\nabla\hat{\phi}_i)^2 + \frac{1}{3!} g \sum_{ijk} Q_{ijk}\hat{\phi}_i\hat{\phi}_j\hat{\phi}_k\right) \tag{3.21}$$

is the renormalised Lagrangian with new masses r_i and coupling constant g, and the counter terms, CT, are given by

$$\mathrm{CT} = \int \mathrm{d}\boldsymbol{x}\Big(\frac{1}{2}\sum_i (Z_i r_{0i} - r_i)\hat{\phi}_{i\mathrm{R}}^2 + \frac{1}{2}\sum_i (Z_i - 1)(\nabla\hat{\phi}_{i\mathrm{R}})^2$$
$$+ \frac{1}{3!}\sum_{ijk} Q_{ijk}(g_0 Z_i^{1/2} Z_j^{1/2} Z_k^{1/2} - g)\hat{\phi}_{i\mathrm{R}}\hat{\phi}_{j\mathrm{R}}\hat{\phi}_{k\mathrm{R}} - \tfrac{1}{2}(Z_\phi Z_{\phi^2} - 1)t\boldsymbol{\phi}_{\mathrm{R}}^2\Big). \tag{3.22}$$

In terms of the fields $\boldsymbol{\phi}$ and ψ, these counter terms are

$$\mathrm{CT} = \int \mathrm{d}\boldsymbol{x}[\tfrac{1}{2}(Z_\phi r_0 - r_\phi)\boldsymbol{\phi}^2 - r_\psi\psi^2 + \tfrac{1}{2}(Z_\phi - 1)(\nabla\boldsymbol{\phi})^2$$
$$+ \tfrac{1}{2}(Z_\psi - 1)(\nabla\psi)^2 + (1/3!)(g_0 Z_\phi Z_\psi^{1/2} - g)\boldsymbol{\phi}^2\psi - \tfrac{1}{2}(Z_\phi Z_{\phi^2} - 1)t\boldsymbol{\phi}^2]. \tag{3.23}$$

If we now consider, when writing down the perturbation expansion for a vertex $\Gamma^{(N,L)}_{i_1\ldots i_N}$, that there is no way to distinguish one ϕ-component from another, but that any ϕ-component contributes differently from ψ, we realise that any iterative procedure to construct counter terms (Amit 1978) would lead us to the counter terms given by (3.23) for the present field theory.

4. Renormalisation group and calculation of the exponents near dimension six

4.1. RG equations and scaling

The renormalisation prescription for the critical theory, equation (3.14), states that as $\varepsilon \to 0$ the renormalised vertex functions are finite, order by order, in the perturbation expansion with w as expansion parameter. This must be true independently of the momentum scale κ, and this condition gives rise to the RG equation for the critical vertex:

$$\Big(\kappa\frac{\partial}{\partial\kappa} + \beta(w)\frac{\partial}{\partial w} - \frac{1}{2}\sum_{r=1}^{N}\gamma_{i_r}(w) + L\gamma_{\phi^2}(w)\Big)\Gamma^{(N,L)}_{i_1\ldots i_N\mathrm{R}}(k_i; p_i; w, \varepsilon, \kappa) = 0, \tag{4.1}$$

where the Wilson functions are defined as

$$\beta(w) = \kappa(\partial w/\partial\kappa)_{w_0} = -\tfrac{1}{2}\varepsilon(\partial \ln w_0/\partial w)^{-1},$$
$$\gamma_{i_r}(w) = \kappa(\partial \ln Z_{i_r}/\partial\kappa)_{w_0} = \beta(w)\partial \ln Z_{i_r}/\partial w,$$
$$\gamma_{\phi^2}(w) = -\kappa(\partial \ln Z_{\phi^2}/\partial\kappa)_{w_0} = -\beta(w)\partial \ln Z_{\phi^2}/\partial w. \tag{4.2}$$

If $\Gamma^{(N,L)}_{i_1\ldots i_N}$ has N_ϕ ϕ-external lines and N_ψ ψ-external lines, (4.1) then reads

$$\left(\kappa\frac{\partial}{\partial\kappa}+\beta(w)\frac{\partial}{\partial w}-\tfrac{1}{2}N_\phi\gamma_\phi(w)-\tfrac{1}{2}N_\psi\gamma_\psi(w)+L\gamma_{\phi^2}(w)\right)\Gamma^{(N,L)}_{i_1\ldots i_N\mathrm{R}}=0, \tag{4.3}$$

where $\gamma_\phi(w)=\beta(w)\partial\ln Z_\phi/\partial w$ and $\gamma_\psi(w)=\beta(w)\partial\ln Z_\psi/\partial w$. For the non-critical functions ($r_0\neq r_{0c}$) we use an expansion of $\Gamma^{(N)}$ in powers of $\delta r_0=r_0-r_{0c}$ with critical $\Gamma^{(N,L)}$'s as coefficients. Since the 'temperature' shift δr_0 applies to ϕ-fields only, ψ being a massless field anyway, we can use the usual expansion (Amit 1978)

$$\Gamma^{(N)}_{i_1\ldots i_N\mathrm{R}}(k_i;t;w,\kappa)=\sum_{L=0}^{\infty}\frac{t^L}{L!}\Gamma^{(N,L)}_{i_1\ldots i_N\mathrm{R}}(k_i;p_i=0;w,\kappa) \tag{4.4}$$

where $t=Z_{\phi^2}^{-1}\delta r_0$.

Consequently, the RG equation for non-critical $\Gamma^{(N)}$'s will be

$$\left(\kappa\frac{\partial}{\partial\kappa}+\beta(w)\frac{\partial}{\partial w}-\tfrac{1}{2}N_\phi\gamma_\phi(w)-\tfrac{1}{2}N_\psi\gamma_\psi(w)+\gamma_{\phi^2}(w)t\frac{\partial}{\partial t}\right)\Gamma^{(N)}_{i_1\ldots i_N\mathrm{R}}(k_i;t,w,\kappa)=0. \tag{4.5}$$

Thus, for pure ϕ-component vertex functions ($N_\psi=0$), the RG equation is the same as for a standard single-field theory. Using Dyson's equation, $\Gamma^{(2)}_{\phi\phi}(k)=[C^{(2)}_{\phi\phi}(k)]^{-1}$, and the well known analysis of the scaling properties of the solution of the RG equation for the case $N_\phi=N=2$, we deduce that the function $C^{(2)}_{\phi\phi}(k)$ satisfies scaling. Namely,

$$\begin{aligned}&C^{(2)}_{\phi\phi}(\rho k)=\rho^{-(2-\eta)}C^{(2)}_{\phi\phi}(k),\\&C^{(2)}_{\phi\phi}(k,t)=k^{-(2-\eta)}f(k\xi),\qquad\text{with }\xi\sim t^{-\nu},\end{aligned} \tag{4.6}$$

where η and ν, given by

$$\eta=\gamma_\phi(w^*),\qquad \nu^{-1}=2-\gamma_{\phi^2}(w^*), \tag{4.7}$$

are the critical exponents and the fixed point is still defined by the condition $\beta(w^*)=0$. All other scaling relations hold in the same way as in the ordinary theory.

4.2. *Calculation of the exponents in the $n=0$ limit to $O(\varepsilon^2)$*

To calculate the ε-expansion of the exponents given in (4.7), we need to know the Wilson functions $\beta(w)$, $\gamma_\phi(w)$, $\gamma_{\phi^2}(w)$, and the renormalisation functions $w_0(w,\varepsilon)$, $Z_\phi(w,\varepsilon)$ and $Z_{\phi^2}(w,\varepsilon)$. We find these by using the method of minimal subtraction of poles for dimensionally regularised vertices.

Since, in the $n=0$ limit,

$$\Gamma^{(2)}_{\psi\psi}(k)=\text{-- -- }\rightarrow\text{ --- --}=k^2$$

is finite, we have

$$Z^{(0)}_\psi(w,\varepsilon)=1. \tag{4.8}$$

Hence, the renormalisation conditions we need are, from (3.14),

$$\begin{aligned}&\Gamma^{(2)}_{\phi\phi\mathrm{R}}(k;w,\varepsilon,\kappa)=Z^{(0)}_\phi\Gamma^{(2)}_{\phi\phi}(k;w_0,\varepsilon,\kappa),\\&\Gamma^{(2,1)}_{\phi\phi\mathrm{R}}(k_1,k_2;w,\varepsilon,\kappa)=Z^{(0)}_\phi Z^{(0)}_{\phi^2}\Gamma^{(2,1)}_{\phi\phi}(k_1,k_2;w_0,\varepsilon,\kappa),\\&\Gamma^{(3)}_{\phi\phi\psi\mathrm{R}}(k_1,k_2;w,\varepsilon,\kappa)=Z^{(0)}_\phi\Gamma^{(3)}_{\phi\phi\psi}(k_1,k_2;w_0,\varepsilon,\kappa).\end{aligned} \tag{4.9}$$

Since bare functions are dimensionally regularised, they contain poles of every order in ε. These must be cancelled, order by order in the perturbation expansion in powers of w, by similar poles in Z_ϕ, Z_{ϕ^2} and in $w_0(w, \varepsilon)$, since left-hand side functions in (4.9) must be finite when $\varepsilon \to 0$. This condition allows us to determine the coefficients of Z_ϕ, Z_{ϕ^2}, w_0 in their expansions in powers of w. As they must be momentum-independent, the calculation of these coefficients provides a careful check on the way the perturbation expansion and the renormalisation of the vertex functions has been done. Also, since the functions β, γ_ϕ and γ_{ϕ^2} must be finite when $\varepsilon \to 0$ and they are given by combinations of w_0, Z_ϕ and Z_{ϕ^2}, we must verify in the calculation of the Wilson functions that higher-order poles in ε do cancel, thus providing another powerful check (Amit 1976, 1978) on the way we operate.

The calculation starts with the expansions for the relevant vertex functions, whose diagrammatic representations were given in figure 1(*a*). To order two-loops these are

$$\begin{aligned}
&\Gamma^{(2)}_{\phi\phi}(k; w_0) = k^2(1 - A_1 w_0^2 - A_2 w_0^4 + \ldots),\\
&\Gamma^{(2,1)}_{\phi\phi}(k_1, k_2; w_0) = 1 + C_1 w_0^2 + C_2 w_0^4 + \ldots,\\
&\Gamma^{(3)}_{\phi\phi\psi}(k_1, k_2; w_0) = \tfrac{1}{3} w_0 \kappa^{\varepsilon/2}(1 + B_1 w_0^2 + B_2 w_0^4 + \ldots),
\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}
&A_1 = \frac{1}{9}\frac{\kappa^2}{k^2} I_1\left(\frac{k}{\kappa}\right), \qquad A_2 = \frac{1}{81}\frac{\kappa^2}{k^2}\left[D_1\left(\frac{k}{\kappa}\right) + D_2\left(\frac{k}{\kappa}\right)\right],\\
&B_1 = C_1 = \tfrac{1}{9} I_2(k_1/\kappa, k_2/\kappa),\\
&B_2 = C_2 = \frac{2}{81} D_3\left(\frac{k_1}{\kappa}, \frac{k_2}{\kappa}\right) + \frac{1}{81} D_4\left(\frac{k_1}{\kappa}, \frac{k_2}{\kappa}\right) + \frac{2}{81} D_5\left(\frac{k_1}{\kappa}, \frac{k_2}{\kappa}\right) + \frac{1}{81} D_6\left(\frac{k_1}{\kappa}, \frac{k_2}{\kappa}\right),
\end{aligned} \tag{4.11}$$

the one-loop (I_1, I_2) and two-loops $(D_1, \ldots, D_6)$ integrals corresponding to the diagrams drawn in figure 1(*a*).

Incidentally, these integrals also appear in Amit's calculation for the Potts model (Amit 1976) and with the same ε-expansions, although there they are associated with topologically different graphs. From (4.10) we deduce that the parameters w_0/w, $Z^{(0)}_\phi$ and $Z^{(0)}_{\phi^2}$ must have odd power coefficients equal to zero, in their expansion in powers of w:

$$\begin{aligned}
&w_0 = w(1 + a_2 w^2 + a_4 w^4 + \ldots),\\
&Z^{(0)}_\phi = 1 + b_2 w^2 + b_4 w^4 + \ldots, \qquad Z^{(0)}_{\phi^2} = 1 + c_2 w^2 + c_4 w^4 + \ldots,
\end{aligned} \tag{4.12}$$

where the a_i's, b_i's and c_i's, like the A_i's, B_i's and C_i's, contain poles of every order in ε. The requirement that poles must cancel, order by order in w, when (4.10) and (4.12) are substituted in (4.9), gives

$$\begin{aligned}
&b_2 = [A_1]_s, \qquad b_4 = [A_2 + (2a_2 + b_2)A_1]_s,\\
&c_2 = -[(b_2 + C_1)]_s, \qquad c_4 = -[C_2 + (2a_2 + b_2 + c_2)C_1 + b_4 + b_2 c_2]_s,\\
&a_2 = -[b_2 + B_1]_s, \qquad a_4 = -[B_2 + (3a_2 + b_2)B_1 + b_4 + a_2 b_2]_s,
\end{aligned} \tag{4.13}$$

where $[A]_s$ means the singular part of A, as a function of ε. We now expand to the

desired order the integrals I_1, I_2 and $D_1, \ldots, D_6$ in their poles in the variable ε, that is

$$I_1(k) = -(k^2/3\varepsilon)(1+\tfrac{7}{12}\varepsilon - \tfrac{1}{2}\varepsilon \ln k^2) + O(\varepsilon),$$
$$I_2(k_1, k_2) = (1/\varepsilon)[1 - \tfrac{3}{4}\varepsilon - L_2(k_1, k_2)\varepsilon] + O(\varepsilon), \tag{4.14}$$

where

$$L_2(k_1, k_2) = \int_0^1 dx \int_0^1 dy\, \theta(1-x-y) \ln [x(1-x)k_1^2 + y(1-y)k_2^2 + 2xyk_1k_2],$$

and

$$D_1(k) = -(1/3\varepsilon^2)k^2(1+\tfrac{3}{2}\varepsilon - \varepsilon \ln k^2) + O(\varepsilon^0),$$
$$D_2(k) = (k^2/18\varepsilon^2)(1+\tfrac{25}{12}\varepsilon - \varepsilon \ln k^2) + O(\varepsilon^0),$$
$$D_3(k_1, k_2) = (1/2\varepsilon^2)(1 - \tfrac{5}{4}\varepsilon - 2\varepsilon L_2(k_1, k_2)) + O(\varepsilon^0),$$
$$D_4(k_1, k_2) = (1/2\varepsilon) + O(\varepsilon^0),$$
$$D_5(k_1, k_2) = -(1/6\varepsilon^2)[1 - \tfrac{11}{12}\varepsilon - 2\varepsilon L_2(k_1, k_2)] + O(\varepsilon^0),$$
$$D_6(k_1, k_2) = (1/2\varepsilon^2)[1 - \tfrac{5}{4}\varepsilon - 2\varepsilon L_2(k_1, k_2)] + O(\varepsilon^0). \tag{4.15}$$

If we substitute these integrals into (4.11), we then verify that coefficients (4.13) do not contain momentum-dependent terms, as these mutually cancel in the calculation. The expansions (4.12), up to the required order, for the renormalisation functions can now be written down explicitly; they are

$$w_0 = w\left[1 - \frac{2}{27\varepsilon} w^2 + \left(\frac{2}{3^5}\frac{1}{\varepsilon^2} - \frac{67}{2^2 \times 3^7}\frac{1}{\varepsilon}\right) w^4 + \ldots\right],$$
$$Z_\phi^{(0)} = 1 - \frac{1}{27\varepsilon} w^2 + \left(\frac{5}{2\times 3^6}\frac{1}{\varepsilon^2} - \frac{13}{2^3 \times 3^7}\frac{1}{\varepsilon}\right) w^4 + \ldots,$$
$$Z_{\phi^2}^{(0)} = 1 - \frac{2}{27\varepsilon} w^2 + \left(\frac{2}{3^5}\frac{1}{\varepsilon^2} - \frac{67}{2^2 \times 3^7}\frac{1}{\varepsilon}\right) w^4 + \ldots. \tag{4.16}$$

Introducing these expansions into the definition (4.2) of the Wilson functions, we find that these last ones have the correct ε-dependence, as cancellations of higher-order poles in their formal expansion coefficients do occur. We have in particular

$$\beta^{(0)}(w) = -\tfrac{1}{2}w(\varepsilon + \tfrac{4}{27} w^2 + (67/3^7)w^4 + \ldots),$$
$$\gamma_\phi^{(0)}(w) = \tfrac{1}{27}w^2 + (13/2^2 \times 3^7)w^4 + \ldots,$$
$$\gamma_{\phi^2}^{(0)}(w) = -\tfrac{2}{27}w^2 - (67/2 \times 3^7)w^4 + \ldots. \tag{4.17}$$

At this point we start looking for a non-trivial fixed point (NTFP) near $\varepsilon = 0$ ($d = 6$), and verify its infrared stability (Amit 1978). Solving

$$\varepsilon + \tfrac{4}{27}w^2 + (67/3^7)w^4 = 0 \qquad \text{for } \varepsilon \to 0,$$

we find a NTFP for

$$(w^*)^2 = -\tfrac{27}{4}\varepsilon - (3^2 \times 67/4^3)\varepsilon^2 \tag{4.18}$$

which is IR stable, as

$$\omega = \beta^{(0)'}(w^*) = \varepsilon - (67/2^4 \times 3)\varepsilon^2 > 0 \qquad \text{for } \varepsilon \to 0. \tag{4.19}$$

The ε-expansion for the critical exponents will then be, to order ε^2,

$$\nu^{-1} = 2 - \tfrac{1}{2}\varepsilon + O(\varepsilon^3), \qquad \eta = -\tfrac{1}{4}\varepsilon - \tfrac{9}{32}\varepsilon^2 + O(\varepsilon^3), \tag{4.20}$$

and this result agrees with the one reported by Pfeuty *et al* (1977). We remark that, as seen in equation (4.18), the fixed point w^* is imaginary as $\varepsilon \to 0$, this being the result of the introduction in § 2 of a fictitious field ψ. Nevertheless, relevant quantities, the exponents ν and η, are real and we will see in appendix 3 that for the physically meaningful Coulomb theory the fixed point $u^* = -\frac{1}{3}(w^*)^2$ is real.

4.3. An exact result for the size exponent ν

The RG method we use allows one to calculate with no effort all the coefficients of the ε-expansion for the size exponent ν, i.e. it provides an exact result for ν as a function of the number of dimensions. It is no accident, in fact, that the ε^2-coefficient in the expansion for ν vanishes, or that to order w^2 the expansion for $Z^{(0)}_{\phi^2}$ is the same as for w_0/w (see equation (4.16)).

Let us consider the $\boldsymbol{\phi}^2\psi$ model in the case of general n. Then, as demonstrated in appendix 2, we can show that, order by order in the expansion parameter, the following relationship exists between the two bare functions $\Gamma^{(3)}_{\phi\phi\psi}$ and $\Gamma^{(2,1)}_{\phi\phi}$:

$$\Gamma^{(3)}_{\phi\phi\psi}(k_1, k_2; w_0) = \tfrac{1}{3} w_0 \kappa^{\varepsilon/2} \Gamma^{(2,1)}_{\phi\phi}(k_1, k_2; w_0). \tag{4.21}$$

To order two-loops, this is particularly evident in equations (4.10) and (4.11). The same relationship must hold for the renormalised vertex functions, so that, using the renormalisation prescription (3.14), we can write

$$\begin{aligned}
&\Gamma^{(3)}_{\phi\phi\psi\mathrm{R}}(k_1, k_2; w) = \tfrac{1}{3} w \kappa^{\varepsilon/2} \Gamma^{(2,1)}_{\phi\phi\mathrm{R}}(k_1, k_2; w) \\
&= (w/w_0) Z_\phi Z_{\phi^2} \tfrac{1}{3} w_0 \kappa^{\varepsilon/2} \Gamma^{(2,1)}_{\phi\phi}(k_1, k_2; w_0) \\
&= (w/w_0) Z_\phi Z_{\phi^2} \Gamma^{(3)}_{\phi\phi}(k_1, k_2; w_0),
\end{aligned} \tag{4.22}$$

from which, having

$$\Gamma^{(3)}_{\phi\phi\psi\mathrm{R}}(k_1, k_2; w) = Z_\phi Z_\psi^{1/2} \Gamma^{(3)}_{\phi\phi\psi}(k_1, k_2; w_0),$$

we find

$$Z_\psi^{1/2} w_0/w = Z_{\phi^2}, \tag{4.23}$$

this relationship being true order by order in the perturbation expansion of both sides.

In particular, in the $n = 0$ case where $Z_\psi = 1$ (see equation (4.8)), equation (4.23) shows that $Z^{(0)}_{\phi^2}$ and w_0/w are not independent parameters, as they must have the same expansion. The most important consequence of (4.23) is however the fact that it provides an exact relation for the exponent ν. If $n \neq 0$, we can verify that, in the same way that $\Gamma^{(2)}_{\phi\phi}(k)$, as solution of a RG equation, satisfies scaling with an exponent η_ϕ given by $\gamma_\phi(w^*)$, $\Gamma^{(2)}_{\psi\psi}(k)$ scales under a change of momentum scale with a critical exponent given by

$$\eta_\psi = \gamma_\psi(w^*), \qquad \text{where } \gamma_\psi(w) = \beta(w)\partial \ln Z_\psi/\partial w. \tag{4.24}$$

The physical meaning of η_ψ for $n \neq 0$ is obscure, being related to the critical behaviour of the Coulomb field $\psi(\boldsymbol{x})$, but for the polyelectrolyte problem $\eta_\psi = 0$ and $\eta_\phi = \eta$, giving $Z^{(0)}_\psi = 1$.

For general n, the result (4.23) gives rise to a relationship involving the functions $\beta(w)$, $\gamma_\psi(w)$ and $\gamma_{\phi^2}(w)$. We have in fact

$$\beta(w)\frac{\partial \ln w_0}{\partial w}=\beta(w)\frac{\partial}{\partial w}\ln\left(wZ_{\phi^2}Z_\psi^{-1/2}\right)=\frac{1}{w}\beta(w)+\beta(w)\frac{\partial \ln Z_{\phi^2}}{\partial w}-\frac{1}{2}\beta(w)\frac{\partial \ln Z_\psi}{\partial w}. \tag{4.25}$$

Remembering now the definitions of $\beta(w)$ and $\gamma_{\phi^2}(w)$ (equation (4.2)) and of $\gamma_\psi(w)$ (equation (4.24)), this leads to the result

$$(1/w)\beta(w)=-\tfrac{1}{2}\varepsilon+\gamma_{\phi^2}(w)+\tfrac{1}{2}\gamma_\psi(w), \tag{4.26}$$

which states that β, γ_{ϕ^2} and γ_ψ are not independent functions. It is useful at this point to observe that in the $n=0$ limit equation (4.26) simply becomes

$$(1/w)\beta^{(0)}(w)=-\tfrac{1}{2}\varepsilon+\gamma^{(0)}_{\phi^2}(w), \tag{4.27}$$

a condition which is verified to order w^2, in the expansions for $\beta^{(0)}$ and $\gamma^{(0)}_{\phi^2}$, in our ε-expansion calculations (see equation (4.17)). The condition for a non-trivial fixed point to occur is now

$$(1/w^*)\beta(w^*)=0, \tag{4.28}$$

and leads to the relationship

$$\gamma_{\phi^2}(w^*)+\tfrac{1}{2}\gamma_\psi(w^*)=\tfrac{1}{2}\varepsilon, \tag{4.29}$$

that is

$$\nu^{-1}-\tfrac{1}{2}\eta_\psi=2-\tfrac{1}{2}\varepsilon=\tfrac{1}{2}(d-2). \tag{4.30}$$

In the $n\neq 0$ case, this would represent a new scaling relation between critical exponents. However, its physical meaning is not at all clear.

In the $n=0$ limit, the formula (4.30) represents instead an exact result for the size exponent of the single polyion:

$$\nu=2/(d-2). \tag{4.31}$$

This is a convenient point at which to stress the fact that the exact result (4.31) is simply a consequence of the particular form of the Lagrangian of the problem, equation (3.1), and of the RG equations satisfied by the correlation functions associated with it. The limit of validity of the result is represented by the existence of the NTFP, equation (4.28), which can be established however only near the critical value of the dimensionality.

5. Conclusion

We have presented the details of the renormalisation of the Lagrangian ϕ^4-field theory interacting with a Coulomb potential. Using the renormalisation approach of t'Hooft and Veltman, the RG treatment leads to a new scaling relation between two critical exponents of the system. In the $n=0$ limit, this gives rise to the exact result $\nu=$

$2/(d-2)$ for the size exponent of the isolated polyelectrolyte chain, in a space whose dimensionality is very close to $d_c = 6$. As first pointed out by Pfeuty *et al* (1977), this result differs from the Flory-type value $\nu = 3/d$ obtainable through self-consistent calculations.

As in many other RG calculations, we are tempted to extrapolate the result $\nu = 2/(d-2)$ to lower dimensionalities, although we cannot verify the existence of a stable fixed point below $d_c = 6$, this being a necessary condition for the result to be true.

The extrapolation procedure proves to be a successful one for the neutral, excluded-volume polymer, for which $d_c = 4$. Extrapolated to $d = 3$, the ε-expansion for the size exponent ν, rigorously true near dimension four, gives the result $\nu \simeq 0.592$ to $O(\varepsilon^2)$ (des Cloizeaux (1975) and reference therein). This result is in good agreement with the most recent and accurate experimental value $\nu = 0.586 \pm 0.004$ (Cotton 1980), which is in turn quite different from the Flory result $\nu = 0.60$.

For the polyelectrolyte chain, the extrapolation seems to be unacceptable below dimension four, when the value $\nu = 1$ is recovered. In three dimensions the radius of gyration would grow as N^2, faster than the chain linear length $L = lN$, as $N \to \infty$, and this is unphysical. Pfeuty *et al* (1977) have provided a phenomenological decimation procedure to show that no finite stable fixed point appears to exist below $d = 4$. These authors also argue that for $d < 4$ the polyelectrolyte chain reaches a new rigid behaviour with $\nu = 1$, although no rigorous support for this quite reasonable result has been given yet.

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Appendix 1. Alternative direct proof of the zero-component field equivalence for an interacting polymer chain

Let us consider a polymer chain of length L interacting with a potential $V(\boldsymbol{x}-\boldsymbol{y})$. The probability distribution function defined in § 2 can be alternatively written as

$$G(\boldsymbol{R}, L, \Lambda) = \frac{1}{Z(L)} \int D\boldsymbol{r}\, \delta(\boldsymbol{R} - \boldsymbol{r}_L + \boldsymbol{r}_0) \exp\left[-\frac{3}{2l} \int_0^L \mathrm{d}s \left(\frac{\mathrm{d}\boldsymbol{r}(s)}{\mathrm{d}s} \right)^2 \right.$$
$$\left. -\frac{\beta}{2l^2} \int_0^L \mathrm{d}s'\, \mathrm{d}s''\, V[\boldsymbol{r}(s') - \boldsymbol{r}(s'')] - \beta \int_0^L \mathrm{d}s\, \Lambda[\boldsymbol{r}(s)] \right], \tag{A1.1}$$

where

$$Z(L) = \int D\boldsymbol{r} \exp\left[-\frac{3}{2l} \int_0^L \mathrm{d}s \left(\frac{\mathrm{d}\boldsymbol{r}(s)}{\mathrm{d}s} \right)^2 \right] \tag{A1.2}$$

is the partition function for the ideal chain, $\beta = (k_B T)^{-1}$ and $\Lambda(\boldsymbol{r})$ is an external potential introduced for mathematical convenience. The physical probability distribution is obtained in the limit $\Lambda(\boldsymbol{r}) = 0$. We can obtain an 'equation of motion' for $G(\boldsymbol{R}, L, \Lambda)$ by considering its change when the length of the chain is increased to $L + \delta L$, with $\delta L \to 0$.

After some algebra, (A1.1) becomes

$$G(\boldsymbol{R}, L+\delta L, \Lambda) = \frac{1}{Z(L+\delta L)} \int Dr\, \mathrm{d}r_{L+\delta L} \delta(\boldsymbol{R} - \boldsymbol{r}_{L+\delta L} + \boldsymbol{r}_0)$$
$$\times \exp\left[-\frac{3}{2l} \int_0^L \mathrm{d}s \left(\frac{\mathrm{d}\boldsymbol{r}(s)}{\mathrm{d}s} \right)^2 - \frac{\beta}{2l^2} \int_0^L \mathrm{d}s'\, \mathrm{d}s'' V(\boldsymbol{r}' - \boldsymbol{r}'') - \beta \int_0^L \mathrm{d}s \Lambda[\boldsymbol{r}(s)] \right]$$
$$\times \exp\left[-\frac{3}{2l} \frac{(\boldsymbol{r}_{L+\delta L} - \boldsymbol{r}_L)^2}{\delta L} - \frac{\beta}{l^2} \delta L \int_0^L \mathrm{d}s\, V[\boldsymbol{r}_L - \boldsymbol{r}(s)] - \beta \delta L \Lambda[\boldsymbol{r}(L)] \right]. \quad \text{(A1.3)}$$

If we put $\boldsymbol{R}' = \boldsymbol{R} - (\boldsymbol{r}_{L+\delta L} - \boldsymbol{r}_L)$, we obtain the following integral equation for G:

$$G(\boldsymbol{R}, L+\delta L, \Lambda) = \frac{Z(L)}{Z(L+\delta L)} \int \mathrm{d}\boldsymbol{R}' \exp\left(-\frac{3}{2l} \frac{(\boldsymbol{R}' - \boldsymbol{R})^2}{\delta L} - \beta \delta L \Lambda(\boldsymbol{r}_0 + \boldsymbol{R}') \right)$$
$$\times G[\boldsymbol{R}', L, \Lambda(\boldsymbol{r}) + (\delta L / l^2) V(\boldsymbol{r}_0 + \boldsymbol{R}' - \boldsymbol{r})], \quad \text{(A1.4)}$$

where

$$\frac{Z(L+\delta L)}{Z(L)} \to \int \mathrm{d}\boldsymbol{R}' \exp\left(-\frac{3}{2l} \frac{(\boldsymbol{R}' - \boldsymbol{R})^2}{\delta L} \right) \qquad \text{as } \delta L \to 0. \quad \text{(A1.5)}$$

Equation (A1.4) is now simplified by the development of both sides to first order in δL. Taking into account that on the right-hand side we must also develop $G(\boldsymbol{R}', L, \Lambda)$ around $\boldsymbol{R}' = \boldsymbol{R}$, we obtain

$$\frac{\partial}{\partial L} G(\boldsymbol{R}, L, \Lambda) = \tfrac{1}{2} b \nabla^2 G(\boldsymbol{R}, L, \Lambda) + \frac{1}{l^2} \int \mathrm{d}\boldsymbol{r} V(\boldsymbol{R} + \boldsymbol{r}_0 - \boldsymbol{r}) \frac{\delta G(\boldsymbol{R}, L, \Lambda)}{\delta \Lambda(\boldsymbol{r})}$$
$$- \beta \Lambda(\boldsymbol{R} + \boldsymbol{r}_0) G(\boldsymbol{R}, L, \Lambda), \quad \text{(A1.6)}$$

where

$$b \delta L = \frac{Z(L)}{Z(L+\delta L)} \frac{1}{d} \int \mathrm{d}\boldsymbol{R}' (\boldsymbol{R}' - \boldsymbol{R})^2 \exp\left(-\frac{3}{2l} \frac{(\boldsymbol{R}' - \boldsymbol{R})^2}{\delta L} \right) = \frac{l}{3} \delta L. \quad \text{(A1.7)}$$

If we put $\boldsymbol{r}_0 = \boldsymbol{r}'$, $\boldsymbol{r}_0 + \boldsymbol{R} = \boldsymbol{r}''$, we can write the equation for the Green function in the form

$$\left(\frac{\partial}{\partial L} + \beta \Lambda(\boldsymbol{r}'') - \frac{l}{6} \nabla^2 \right) G(\boldsymbol{r}'', \boldsymbol{r}', L, \Lambda) - \frac{1}{l^2} \int \mathrm{d}\boldsymbol{r} V(\boldsymbol{r}'' - \boldsymbol{r}) \frac{\delta G(\boldsymbol{r}'', \boldsymbol{r}', L, \Lambda)}{\delta \Lambda(\boldsymbol{r})} = 0, \quad \text{(A1.8)}$$

subject to the boundary condition

$$\lim_{L \to 0} G(\boldsymbol{R}, L, \Lambda) = \delta(\boldsymbol{R}). \quad \text{(A1.9)}$$

This equation is now in a standard form (cf Kadanoff and Baym 1962, equation (5.12)).

We show explicitly that it is the equation satisfied by the Green function of a zero-component $\boldsymbol{\phi}^4$- field theory with interaction $V(\boldsymbol{x} - \boldsymbol{y})$.

First, we introduce the Laplace transform defined by equation (2.8). As a result of equations (A1.8) and (A1.9) it satisfies

$$\left(\frac{s}{l} + \beta \Lambda(\boldsymbol{r}'') - \frac{l}{6} \nabla^2 \right) G(\boldsymbol{r}'', \boldsymbol{r}', s, \Lambda) - \frac{1}{l^2} \int \mathrm{d}\boldsymbol{r} V(\boldsymbol{r}'' - \boldsymbol{r}) \frac{\delta G(\boldsymbol{r}'', \boldsymbol{r}', s, \Lambda)}{\delta \Lambda(\boldsymbol{r})} = (1/l) \delta(\boldsymbol{r}'' - \boldsymbol{r}'). \quad \text{(A1.10)}$$

From this equation it is possible to expand G in powers of the interaction V to obtain the usual expansion, but we choose to compare the equation directly with that derived from a $\boldsymbol{\phi}^4$-field theory.

The Lagrangian for the $\boldsymbol{\phi}^4$-field theory with an external potential $\Lambda(\boldsymbol{r})$ is

$$\mathscr{L} = \int d\boldsymbol{r}\{\tfrac{1}{2}|\nabla\boldsymbol{\phi}|^2 + \tfrac{1}{2}[r_0 + \Lambda(\boldsymbol{r})]|\boldsymbol{\phi}|^2\} + \frac{1}{4!}\int d\boldsymbol{r}'\, d\boldsymbol{r}'' u(\boldsymbol{r}'-\boldsymbol{r}'')|\boldsymbol{\phi}(\boldsymbol{r}')|^2|\boldsymbol{\phi}(\boldsymbol{r}'')|^2, \tag{A1.11}$$

where

$$|\boldsymbol{\phi}(\boldsymbol{r})|^2 = \sum_{i=1}^{n}[\phi_i(\boldsymbol{r})]^2, \qquad |\nabla\boldsymbol{\phi}|^2 = \sum_{i=1}^{n}(\nabla\phi_i)^2. \tag{A1.12}$$

The corresponding Green function is

$$G_{ij}(\boldsymbol{r}, \boldsymbol{r}', \Lambda) = Z^{-1}\int D\boldsymbol{\phi}\, \exp(-\mathscr{L})\phi_i(\boldsymbol{r})\phi_j(\boldsymbol{r}'), \tag{A1.13}$$

where

$$Z = \int D\boldsymbol{\phi}\, \exp(-\mathscr{L}) \tag{A1.14}$$

and the integration is over all values of $\boldsymbol{\phi}(\boldsymbol{r})$ at each field point $\boldsymbol{r}$. To obtain the equation for the Green function, consider the quantity Q_{ij} defined by

$$Q_{ij}(\boldsymbol{r}, \boldsymbol{r}') = Z^{-1}\int D\boldsymbol{\phi}\, \phi_j(\boldsymbol{r}')\frac{\delta}{\delta\phi_i(\boldsymbol{r})}\exp(-\mathscr{L}[\boldsymbol{\phi}]). \tag{A1.15}$$

Direct evaluation yields

$$\begin{aligned} Q_{ij}(\boldsymbol{r}, \boldsymbol{r}') &= -Z^{-1}\int D\boldsymbol{\phi}\, \phi_j(\boldsymbol{r}')\Big(-\nabla^2\phi_i(\boldsymbol{r}) + [r_0 + \Lambda(\boldsymbol{r})]\phi_i(\boldsymbol{r}) \\ &\qquad + \frac{1}{3!}\int d\boldsymbol{r}'' u(\boldsymbol{r}-\boldsymbol{r}'')\phi_i(\boldsymbol{r})|\boldsymbol{\phi}(\boldsymbol{r}'')|^2\Big)\exp(-\mathscr{L}) \\ &= -[r_0 + \Lambda(\boldsymbol{r}) - \nabla^2]G_{ij}(\boldsymbol{r}, \boldsymbol{r}', \Lambda) \\ &\qquad - \frac{Z^{-1}}{3!}\int D\boldsymbol{\phi}\, \phi_j(\boldsymbol{r}')\int d\boldsymbol{r}'' u(\boldsymbol{r}-\boldsymbol{r}'')\phi_i(\boldsymbol{r})|\boldsymbol{\phi}(\boldsymbol{r}'')|^2\exp(-\mathscr{L}). \end{aligned} \tag{A1.16}$$

However,

$$\begin{aligned} \frac{\delta}{\delta\Lambda(\boldsymbol{r}'')}G_{ij}(\boldsymbol{r}, \boldsymbol{r}', \Lambda) &= -\tfrac{1}{2}Z^{-1}\int D\boldsymbol{\phi}\, \phi_j(\boldsymbol{r}')|\boldsymbol{\phi}(\boldsymbol{r}'')|^2\phi_i(\boldsymbol{r})\exp(-\mathscr{L}) \\ &\qquad + \tfrac{1}{2}G_{ij}(\boldsymbol{r}, \boldsymbol{r}', \Lambda)\sum_l G_{ll}(\boldsymbol{r}'', \boldsymbol{r}'', \Lambda) \end{aligned} \tag{A1.17}$$

where the last term comes from the differentiation of Z. From equations (A1.16) and (A1.17) we see that

$$\begin{aligned} Q_{ij}(\boldsymbol{r}, \boldsymbol{r}') &= -[r_0 + \Lambda(\boldsymbol{r}) - \nabla^2]G_{ij}(\boldsymbol{r}, \boldsymbol{r}', \Lambda) \\ &\qquad + \frac{1}{3!}\int d\boldsymbol{r}'' u(\boldsymbol{r}-\boldsymbol{r}'')\Big(2\frac{\delta}{\delta\Lambda(\boldsymbol{r}'')}G_{ij}(\boldsymbol{r}, \boldsymbol{r}', \Lambda) - G_{ij}(\boldsymbol{r}, \boldsymbol{r}', \Lambda)\sum_l G_{ll}(\boldsymbol{r}'', \boldsymbol{r}'', \Lambda)\Big). \end{aligned} \tag{A1.18}$$

However, Q_{ij} can also be evaluated by integrating equation (A1.15) by parts over the variable $\phi_i(\boldsymbol{r})$. This leads to

$$Q_{ij}(\boldsymbol{r}, \boldsymbol{r}') = -\delta_{ij}\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{A1.19}$$

If we note, too, that

$$G_{ij}(\boldsymbol{r}, \boldsymbol{r}', \Lambda) = \delta_{ij}G(\boldsymbol{r}, \boldsymbol{r}', \Lambda), \tag{A1.20}$$

we obtain the equation for the Green function,

$$[-\nabla^2 + r_0 + \Lambda(\boldsymbol{r})]G(\boldsymbol{r}, \boldsymbol{r}', \Lambda) - \frac{1}{3!}\int d\boldsymbol{r}'' u(\boldsymbol{r}-\boldsymbol{r}'')\left(2\frac{\delta G(\boldsymbol{r}, \boldsymbol{r}', \Lambda)}{\delta\Lambda(\boldsymbol{r}'')} - nG(\boldsymbol{r}'', \boldsymbol{r}'', \Lambda)G(\boldsymbol{r}, \boldsymbol{r}', \Lambda)\right)$$
$$= \delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{A1.21}$$

With $n = 0$ and a scaling and proper identification of the variables, this equation is identical to (A1.10) and shows that the polymer problem is equivalent to a zero-component ϕ^4-field theory.

Appendix 2. Proof of the relationship $\Gamma^{(3)}_{\phi\phi\psi} = \frac{1}{3}g_0\Gamma^{(2,1)}_{\phi\phi}$

The relationship (4.21) between $\Gamma^{(3)}_{\phi\phi\psi}$ and $\Gamma^{(2,1)}_{\phi\phi}$ is at the origin of the exact result found for the exponent ν.

It can be proved to be true for all values of momenta k_1 and k_2 by making use of diagrammatic perturbation theory. This turns out to be a lengthy and rather involved proof, since for every diagram of $\Gamma^{(3)}_{\phi\phi\psi}$ and of $\Gamma^{(2,1)}_{\phi\phi}$ having the same topological form we have to prove that all factors, in particular symmetry and combinatorial ones, match properly.

A more direct proof can be given by exploiting the form of the Lagrangian (2.17), although the method holds for $k_1 + k_2 = 0$ only. Our task is to prove the following two relationships for general n:

$$\Gamma^{(3)}_{\phi\phi\psi}(k, -k, 0; r_0, g_0) = \frac{1}{3}g_0\frac{\partial}{\partial r_0}\Gamma^{(2)}_{\phi\phi}(k, -k; r_0, g_0), \tag{A2.1}$$

$$\Gamma^{(2,1)}_{\phi\phi}(k, -k, 0; r_0, g_0) = \frac{\partial}{\partial r_0}\Gamma^{(2)}_{\phi\phi}(k, -k; r_0, g_0), \tag{A2.2}$$

since from these, equation (4.21) immediately follows.

Equation (A2.2) is actually true for any massive field theory with a term of the form

$$\tfrac{1}{2}r_0\int d\boldsymbol{x}[\boldsymbol{\phi}(\boldsymbol{x})]^2$$

in the Lagrangian.

With notations from § 3.1, we have in fact

$$\frac{\partial}{\partial r_0}\Gamma^{(2)}_{\phi\phi}(k, -k) = \frac{\delta^2}{\delta\bar{\phi}(k)\delta\bar{\phi}(-k)}\left(\frac{\partial}{\partial r_0}\Gamma[\bar{\phi}, \bar{\psi}, t]\right)\Big|_0, \tag{A2.3}$$

where, for the sake of simplicity, ϕ is considered scalar (although everything holds for general n) and the symbol $|_0$ means that functional derivatives are evaluated for the equilibrium values of $\bar{\phi}$, $\bar{\psi}$, and t. Differentiating now both sides of equation (3.5) with

respect to r_0, but keeping $\bar{\phi}$ and $\bar{\psi}$ fixed, we can easily establish

$$\frac{\partial}{\partial r_0}\Gamma[\bar{\phi},\bar{\psi},t]=-\frac{\partial}{\partial r_0}F[h,j,t]. \tag{A2.4}$$

The functional F is now defined as $F=\ln Z$, where

$$Z[h,j,t]=\frac{1}{Z_0}\int D\hat{\phi}\,\exp(-L-S), \qquad Z_0=\int D\hat{\phi}\,\exp(-L), \tag{A2.5}$$

L being the Lagrangian of the field, S the source term (3.4) and $\hat{\phi}$ indicating the totality of fields. From (A2.4) and (A2.5) we have

$$-\frac{\partial}{\partial r_0}\Gamma[\bar{\phi},\bar{\psi},t]=\frac{1}{Z[h,j,t]}\frac{1}{Z_0}\int D\hat{\phi}[-\tfrac{1}{2}\phi^2(0)]\exp(-L-S)$$

$$-\frac{1}{Z_0}\int D\hat{\phi}[-\tfrac{1}{2}\phi^2(0)]\exp(-L)=-\frac{\delta F}{\delta t(0)}+(\tfrac{1}{2}\phi^2(0))^0,$$

that is

$$\partial\Gamma/\partial r_0=\delta\Gamma/\delta t(0)-(\tfrac{1}{2}\phi^2(0))^0, \tag{A2.6}$$

where $t(0)=t(k=0)$, $\phi^2(0)=\phi^2(k=0)$ and where $\phi^2(k)$ is the Fourier transform of $[\phi(x)]^2$. Introducing (A2.6) into (A2.3), we have

$$\frac{\partial}{\partial r_0}\Gamma^{(2)}_{\phi\phi}(k,-k)=\frac{\delta^3\Gamma}{\delta\bar{\phi}(k)\delta\bar{\phi}(-k)\delta t(0)}\bigg|_0\equiv\Gamma^{(2,1)}_{\phi\phi}(k,-k,0),$$

which proves (A2.2).

To prove (A2.1) we take a Lagrangian L of the form (2.17) with source term S given by (3.4). We then write, as in (A2.3),

$$\frac{1}{3}g_0\frac{\partial}{\partial r_0}\Gamma^{(2)}_{\phi\phi}(k,-k)=\frac{\delta^2}{\delta\bar{\phi}(k)\delta\bar{\phi}(-k)}\left(\frac{1}{3}g_0\frac{\partial}{\partial r_0}\Gamma[\bar{\phi},\bar{\psi},t]\right)\bigg|_0. \tag{A2.7}$$

Proceeding as before, we find that

$$\frac{1}{3}g_0\frac{\partial}{\partial r_0}\Gamma=-\frac{1}{3}g_0\frac{\partial}{\partial r_0}F$$

$$=\frac{1}{Z[h,j,t]}\frac{1}{Z_0}\int D\hat{\phi}\left(\frac{1}{3!}g_0\phi^2(0)\right)\exp(-L-S)$$

$$-\frac{1}{Z_0}\int D\hat{\phi}\left(\frac{1}{3!}g_0\phi^2(0)\right)\exp(-L). \tag{A2.8}$$

Because of the form of Lagrangian (2.17), we can write down the following equation:

$$0\equiv\int D\hat{\phi}\frac{\delta}{\delta\psi(k)}\exp(-L-S)$$

$$=\int D\hat{\phi}\left(-k^2\psi(-k)-\frac{1}{3!}g_0\phi^2(-k)+j(-k)\right)\exp(-L-S), \tag{A2.9}$$

which inserted into (A2.8) for $k=0$, gives

$$\tfrac{1}{3}g_0\partial\Gamma/\partial r_0=j(0)=\delta\Gamma/\delta\bar{\psi}(0) \tag{A2.10}$$

and, from (A2.7), we finally obtain

$$\frac{1}{3} g_0 \frac{\partial}{\partial r_0} \Gamma^{(2)}_{\phi\phi}(k,-k) = \frac{\delta^3 \Gamma}{\delta\bar{\phi}(k)\delta\bar{\phi}(-k)\delta\bar{\psi}(0)}\bigg|_0 \equiv \Gamma^{(3)}_{\phi\phi\psi}(k,-k,0).$$

Appendix 3. Renormalisation and calculation of the exponents for the Coulomb theory

We briefly present here the renormalisation of the Coulomb field theory and the deduction of its critical exponents in the $n = 0$ limit. Beside the fact that the theory is non-trivially renormalisable, the calculation of the critical exponents provides further checks on the RG procedure we use, confirms the existence of a real NTFP near dimension six and gives results in agreement with the calculation of § 4.

Green functions are now defined for the components of one single field $\boldsymbol{\phi}$ and perturbation expansions for them are represented in terms of Feynman graphs in the usual way. For the Coulomb theory the interaction vertex is drawn as a dotted line, representing the factor $v(\boldsymbol{k})$, with two pairs of full lines carrying momentum and spin labels.

Since ψ is a fictitious field, it turns out that Green functions involving ϕ-components only must have the same perturbation expansion in both formalisms, when the relationship $g_0^2 = -3\lambda_0$ is taken into account. For the same reason, diagrams must be topologically identical in both theories, although in the present case dotted lines represent interactions rather than free-field propagators.

In principle, expansions for the 1PI parts of the Green functions involved change when the Coulomb theory is considered, as these vertex functions are now defined with respect to ϕ-body cuts only. This is because 1PI parts are generated from a Legendre transform of F which is referred to $\bar{\phi}$-components only (see equation (3.5)). Consequently, as for any non-point-interacting theory, we have 1PI parts which are not reducible with respect to their interaction lines. This feature generates a non-trivially renormalisable theory.

A3.1. Renormalisation of the theory

The same kind of primitive UV divergences analysis as in § 3.2 allows us to conclude that

$$\delta(\text{UV}) = -r(6-d) + (d + E - \tfrac{1}{2}Ed), \tag{A3.1}$$

from which we deduce that we still have $d_c = 6$, hence $\varepsilon = 6 - d$, and that the relevant primitively divergent vertices for $\varepsilon \geqslant 0$ are $\Omega^{(2)}$ and $\Omega^{(2,1)}$ (by $\Omega^{(N,L)}$ we now denote the 1PI parts of a Green function $G^{(N,L)}$, with respect to ϕ-body cuts only). However, it is not difficult to see that also the reducible part $\hat{\Omega}^{(4)}$ of the four-point vertex $\Omega^{(4)}$ diverges as $\Lambda \to \infty$, thus providing the third renormalisation condition.

As can be seen from figure 2, where the $n = 0$ diagrammatic expansion for $\Omega^{(4)}$ is given to order two-loops, the UV divergence of $\hat{\Omega}^{(4)}$ is due to the interaction line that makes its diagrams one-particle-reducible by not carrying any internal momentum. This feature is not found in (A3.1) where it is supposed that every interaction line carries an internal momentum.

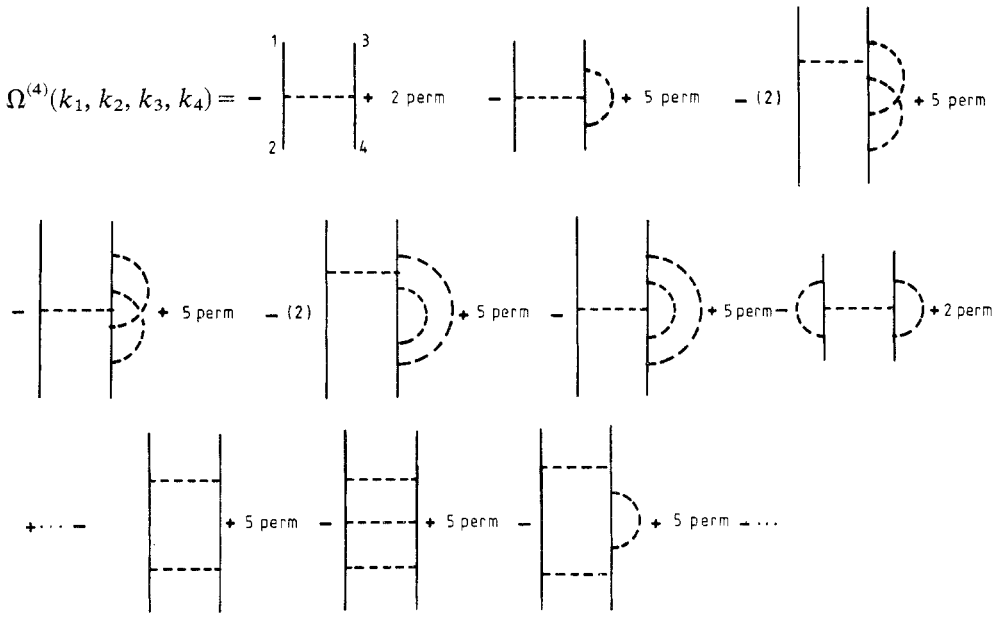
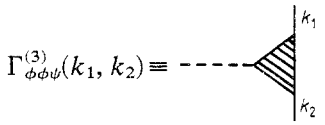


Figure 2. Diagrams contributing to $\Omega^{(4)}$ to order two-loops for $n = 0$. Those contributing to $\hat{\Omega}^{(4)}$ are drawn in the first two lines.

To find out the correct renormalisation procedure for the present theory, we will use what we have learnt about the related $\phi^2\psi$ theory. We first observe that we must have, for general n,

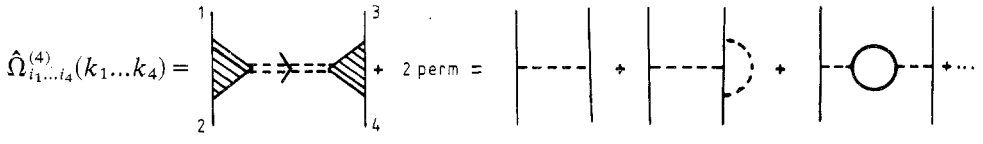
$$\Omega^{(2)}(k) = \Gamma^{(2)}_{\phi\phi}(k), \qquad \Omega^{(2,1)}(k_1, k_2) = \Gamma^{(2,1)}_{\phi\phi}(k_1, k_2), \tag{A3.2}$$

since one-particle-reducible (with respect to ψ-body cuts) diagrams in $\Omega^{(2)}$ and $\Omega^{(2,1)}$ do not contribute. Furthermore, if



(A3.3)

then we have, for general n,



(A3.4)

which we can write in analytic form as

$$\hat{\Omega}^{(4)}_{i_1 \ldots i_4}(k_1 \ldots k_4) = \Gamma^{(3)}_{\phi_{i_1}\phi_{i_2}\psi}(k_1, k_2) C^{(2)}_{\psi\psi}(k_1 + k_2) \Gamma^{(3)}_{\phi_{i_3}\phi_{i_4}\psi}(k_3, k_4) + 2 \text{ permutations.} \tag{A3.5}$$

We know from § 3.3 how to renomalise the vertices $\Gamma^{(2)}_{\phi\phi}$, $\Gamma^{(2,1)}_{\phi\phi}$ and $\Gamma^{(3)}_{\phi\phi\psi}$ for general n.

Observing that

$$C^{(2)}_{\psi\psi R}(k) = Z_\psi^{-1} C^{(2)}_{\psi\psi}(k), \tag{A3.6}$$

we are now ready to write down the renormalisation prescription for the relevant vertices $\Omega^{(2)}$, $\Omega^{(2,1)}$ and $\hat{\Omega}^{(4)}$.

There are three renormalisation parameters, $u_0(u, \varepsilon)$, $Z_\phi(u, \varepsilon)$ and $Z_{\phi^2}(u, \varepsilon)$, where u_0 and u are dimensionless coupling constants defined by

$$u_0 = \lambda_0 \kappa^{-\varepsilon}, \qquad u = \lambda \kappa^{-\varepsilon}. \tag{A3.7}$$

These parameters must be the same as those defined in § 3, once the transformation $w_0^2 = -3u_0$, $w^2 = -3u$ has been performed. As expected, the parameter Z_ψ does not intervene and we find

$$\begin{aligned}
&\Omega^{(2)}_R(k; u, \varepsilon, \kappa) = Z_\phi \Omega^{(2)}(k; \lambda_0), \\
&\Omega^{(2,1)}_R(k_1, k_2; u, \varepsilon, \kappa) = Z_\phi Z_\phi^2 \Omega^{(2,1)}(k_1, k_2; \lambda_0), \\
&\hat{\Omega}^{(4)}_{i_1 \ldots i_4 R}(k_1 \ldots k_4; u, \varepsilon, \kappa) = Z_{\phi^2} \hat{\Omega}^{(4)}_{i_1 \ldots i_4}(k_1 \ldots k_4; \lambda_0),
\end{aligned} \tag{A3.8}$$

which looks rather like the ordinary renormalisation procedure for the point-interacting ϕ^4-theory, except that now only the reducible part $\hat{\Omega}^{(N)}$ has to be renormalised. This can be seen for $N > 4$.

A3.2. RG and calculation of the exponents to $O(\varepsilon^2)$

From the given renormalisation prescription we deduce that a RG equation can be written down both for the critical ($r_0 = r_{0c}$ or $r = 0$) and non-critical vertices. In particular we deduce that $\Omega^{(2)}$ satisfies scaling properties and critical exponents can be defined in the usual way. The limit $n = 0$ is here considered in the calculation of the exponents η and ν to $O(\varepsilon^2)$. We need to know the first terms in the expansions for the Wilson functions, here defined as

$$\begin{aligned}
&\beta^{(0)}(u) = \kappa(\partial u/\partial \kappa)_{u_0} = -\varepsilon(\partial \ln u_0/\partial u)^{-1}, \\
&\gamma^{(0)}_\phi(u) = \beta^{(0)}(u) \partial \ln Z^{(0)}_\phi/\partial u, \qquad \gamma^{(0)}_{\phi^2}(u) = -\beta^{(0)}(u) \partial \ln Z^{(0)}_{\phi^2}/\partial u,
\end{aligned} \tag{A3.9}$$

so that we will have for the exponents: $\eta = \gamma^{(0)}_\phi(u^*)$, $\nu^{-1} = 2 - \gamma^{(0)}_{\phi^2}(u^*)$ where $u^* = \text{NTFP}$ for $\beta^{(0)}(u)$.

The calculation proceeds within the context of minimal subtraction of poles from conditions (A3.8) with bare functions dimensionally regularised.

To order two-loops we have

$$\begin{aligned}
&\Omega^{(2)}(k) = k^2(1 + u_0 A_1 - u_0^2 A_2 + \ldots), \\
&\Omega^{(2,1)}(k_1, k_2) = 1 - u_0 C_1 + u_0^2 C_2 + \ldots, \\
&\hat{\Omega}^{(4)}(k_i) = u_0 \kappa^\varepsilon (B_1 - u_0 B_2 + u_0^2 B_3 + \ldots),
\end{aligned} \tag{A3.10}$$

corresponding to the diagrams drawn in figure 2 for $\hat{\Omega}^{(4)}$ and to the diagrams in figure 1(*a*), opportunely re-evaluated for the present theory, for $\Omega^{(2)}$ and $\Omega^{(2,1)}$. The coefficients A_i, B_i and C_i are in this case given by

$$A_1 = \frac{1}{3} \frac{\kappa^2}{k^2} I_1\left(\frac{k}{\kappa}\right), \qquad A_2 = \frac{1}{9} \frac{\kappa^2}{k^2}\left[D_1\left(\frac{k}{\kappa}\right) + D_2\left(\frac{k}{\kappa}\right)\right],$$

$$C_1 = \frac{1}{3} I_2\left(\frac{k_1}{\kappa}, \frac{k_2}{\kappa}\right),$$

$$C_2 = \frac{2}{9} D_3\left(\frac{k_1}{\kappa}, \frac{k_2}{\kappa}\right) + \frac{1}{9} D_4\left(\frac{k_1}{\kappa}, \frac{k_2}{\kappa}\right) + \frac{2}{9} D_5\left(\frac{k_1}{\kappa}, \frac{k_2}{\kappa}\right) + \frac{1}{9} D_6\left(\frac{k_1}{\kappa}, \frac{k_2}{\kappa}\right),$$

$$B_1 = \frac{1}{3} \frac{1}{(k_1+k_2)^2} + 2 \text{ perm},$$

$$B_2 = \frac{1}{9} \frac{1}{(k_1+k_2)^2} I_2\left(\frac{k_3}{\kappa}, \frac{k_1+k_2}{\kappa}\right) + 5 \text{ perm},$$

$$B_3 = \frac{1}{27} \frac{1}{(k_1+k_2)^2} D_3\left(\frac{k_3}{\kappa}, \frac{k_1+k_2}{\kappa}\right) + 11 \text{ perm} + \frac{1}{27} \frac{1}{(k_1+k_2)^2}$$
$$\times D_4\left(\frac{k_3}{\kappa}, \frac{k_1+k_2}{\kappa}\right) + 5 \text{ perm} + \frac{1}{27} \frac{1}{(k_1+k_2)^2} D_5\left(\frac{k_3}{\kappa}, \frac{k_1+k_2}{\kappa}\right) + 11 \text{ perm}$$
$$+ \frac{1}{27} \frac{1}{(k_1+k_2)^2} D_6\left(\frac{k_3}{\kappa}, \frac{k_1+k_2}{\kappa}\right) + 5 \text{ perm} + \frac{1}{27} \frac{1}{(k_1+k_2)^2}$$
$$\times I_2\left(\frac{k_1}{\kappa}, -\frac{k_1+k_2}{\kappa}\right) I_2\left(\frac{k_3}{\kappa}, \frac{k_1+k_2}{\kappa}\right) + 2 \text{ perm} \qquad \text{(A3.11)}$$

where the one-loop integrals I_1, I_2 and the two-loop integrals $D_1, \ldots, D_6$ are the same as in § 3.

The three renormalisation parameters are now expanded as power series in u:

$$u_0 = u(1 + a_1 u + a_2 u^2 + \ldots),$$
$$Z_\phi^{(0)} = 1 + b_1 u + b_2 u^2 + \ldots,$$
$$Z_{\phi^2}^{(0)} = 1 + c_1 u + c_2 u^2 + \ldots, \qquad \text{(A3.12)}$$

where the a_i, b_i and c_i are determined by the requirement of minimal subtraction of poles. We have, with usual notations,

$$b_1 = [-A_1]_s, \qquad b_2 = [A_2 - A_1(a_1 + b_1)]_s,$$
$$c_1 = [-b_1 + C_1]_s, \qquad c_2 = [(a_1 + b_1 + c_1)C_1 - C_2 - b_2 - b_1 c_1]_s,$$
$$a_1 = [-2b_1 + B_2/B_1]_s, \qquad a_2 = \left[\frac{2(a_1+b_1)B_2 - B_3}{B_1} - 2b_2 - 2b_1 a_1 - b_1^2\right]_s. \qquad \text{(A3.13)}$$

Introducing for the A_i, B_i and C_i their expressions (A3.11) in terms of momentum integrals, and for these their ε-expansions reported in equations (4.14) and (4.15), we can verify that momentum-dependent terms, when all permutations are taken into account, do cancel. The renormalisation parameters (A3.12) are then given by

$$u_0 = u\left[1 + \frac{4}{9\varepsilon} u + \left(\frac{16}{81} \frac{1}{\varepsilon^2} - \frac{67}{6 \times 81} \frac{1}{\varepsilon}\right) u^2 + \ldots\right],$$

$$Z_\phi^{(0)} = 1 + \frac{1}{9\varepsilon} u + \left(\frac{5}{9 \times 18} \frac{1}{\varepsilon^2} - \frac{13}{9 \times 12 \times 18} \frac{1}{\varepsilon}\right) u^2 + \ldots,$$

$$Z_{\phi^2}^{(0)} = 1 + \frac{2}{9\varepsilon} u + \left(\frac{2}{27} \frac{1}{\varepsilon^2} - \frac{67}{12 \times 81} \frac{1}{\varepsilon}\right) u^2 + \ldots. \qquad \text{(A3.14)}$$

We remark at this point that no relationship can be observed between the expansion for $u_0(u, \varepsilon)$ and $Z^{(0)}_{\phi^2}(u, \varepsilon)$. Nevertheless, once the transformation $u_0 = -\frac{1}{3}w_0^2$ and $u = -\frac{1}{3}w^2$ is performed, we recover the expansions (4.16) for the functions $w_0(w, \varepsilon)$, $Z^{(0)}_{\phi}(w, \varepsilon)$ and $Z^{(0)}_{\phi^2}(w, \varepsilon)$, and the relationship $Z^{(0)}_{\phi^2} = w_0/w$ is observed. Finally, calculating the Wilson functions (A3.9) from expansions (A3.14), we check once again that higher-order poles in ε cancel and we obtain the expansions

$$\beta^{(0)}(u) = -u(\varepsilon - \tfrac{4}{9}u + (67/3\times 81)u^2 + \ldots),$$
$$\gamma^{(0)}_{\phi}(u) = -\tfrac{1}{9}u + (13/12\times 81)u^2 + \ldots,$$
$$\gamma^{(0)}_{\phi^2}(u) = \tfrac{2}{9}u - (67/6\times 81)u^2 + \ldots. \qquad \text{(A3.15)}$$

These are in turn converted into the corresponding functions (4.17) after transformation of the coupling constant, making the present calculation consistent with the one in § 4. The interesting result is that $\beta^{(0)}(u)$ has a real NTFP near $d = 6$ given by, to order ε^2,

$$u^* = \tfrac{9}{4}\varepsilon + (3\times 67/4^3)\varepsilon^2 + \ldots. \qquad \text{(A3.16)}$$

Again, we can check that $u^* = -\frac{1}{3}w^{*2}$, with w^* given by (4.18); moreover, the fixed point is infrared stable, being

$$\omega = \beta^{(0)\prime}(u^*) = \varepsilon - \tfrac{67}{48}\varepsilon^2 + \ldots > 0 \qquad \text{when } \varepsilon \to 0. \qquad \text{(A3.17)}$$

Finally, the critical exponents, by substitution of u^* into (A3.15), are

$$\eta = -\tfrac{1}{4}\varepsilon - \tfrac{9}{32}\varepsilon^2 + O(\varepsilon^3), \qquad \nu^{-1} = 2 - \tfrac{1}{2}\varepsilon + O(\varepsilon^3), \qquad \text{(A3.18)}$$

in agreement with (4.20).

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